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MAXIMAL ORDERS OVER REGULAR LOCAL RINGS(1)

BY
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Abstract. In this paper various sufficient conditions are given for the maximality of an R-order in a finite-dimensional central simple K-algebra, where R is a regular local ring whose quotient field is K. Stronger results are obtained when we assume the dimension of R to be three. This work depends upon earlier results of this author [5] for regular local rings of dimension two, and the fundamental work of Auslander and Goldman [1] for dimension one.

Introduction. Let $(R, \mathfrak{M})$ be a regular local ring with maximal ideal $\mathfrak{M}$, and K the quotient field of R. Let Σ be a central simple K-algebra, finite dimensional over K, and Λ an R-order in Σ. Under what conditions is Λ maximal? Auslander and Goldman [1] have shown that if gl.dim. $R=1$ then Λ is maximal if and only if gl.dim. $\Lambda=1$ and Λ is quasi-local (i.e., Λ has a unique maximal two-sided ideal, necessarily Rad Λ, the Jacobson radical of Λ). In [5] this author has shown that when gl.dim. $R=2$, a sufficient condition for maximality is that (a) gl.dim. $\Lambda<\infty$ and (b) Λ is quasi-local [5, Theorem 5.4]. (Under these circumstances gl.dim. $\Lambda=2$.) Examples were given to show that neither (a) nor (b) is necessary. The natural generalization of this theorem is the following:

If gl.dim. $R=n$, gl.dim. $\Lambda<\infty$, and Λ is quasi-local, then Λ is maximal. (*Note.* These conditions ensure that gl.dim. $\Lambda=n$ and Λ is R-free [5, Corollary 2.17].)

In this paper we prove a weaker result. Namely, we prove the above statement under the additional hypothesis that Λ is contained in an R-free maximal order. It is still an open question whether Σ must contain *any* R-free maximal order (see [1, p. 20]). We obtain a slightly stronger result in dimension three. There we show that if Σ does contain an R-free maximal order Γ (not necessarily containing Λ) then Λ is isomorphic to Γ. Technical difficulties prevent us from extending this result to higher dimensions.

Two other theorems are proved for dimension three which do not hypothesize the existence of R-free maximal orders and which are therefore of more practical value in determining the maximality of a given Λ. The first requires instead that $\mathfrak{M}\not\subseteq N^2$, where $N=\text{Rad }\Lambda$. The second requires the existence of an element $x \in \mathfrak{M}-\mathfrak{M}^2$ such that the center of $\Lambda/x\Lambda$ is R/x and gl.dim. $\Lambda/x\Lambda<\infty$. An example

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is given in which the second, but not the first theorem, is applicable. Both these theorems follow from a more general one, valid for regular local rings of any dimension: Suppose Λ is an R-reflexive order and $x \in \mathfrak{M}-\mathfrak{M}^2$ is such that $\Lambda_{(x)}$ is $R_{(x)}$-separable. If $\Lambda/x\Lambda$ is a maximal R/x-order in $\Lambda_{(x)}/x\Lambda_{(x)}$, then Λ is a maximal R-order.

In a sense, the maximality of Λ really boils down to the question of whether $\Lambda \otimes_R R_{\mathfrak{P}}$ is quasi-local for every height one prime $\mathfrak{P}$ in R. For Λ is maximal if and only if Λ is R-reflexive and $\Lambda \otimes_R R_{\mathfrak{P}}$ is a maximal $R_{\mathfrak{P}}$-order for every height one prime $\mathfrak{P}$ [1, Theorem 1.5]. If gl.dim. $\Lambda<\infty$, then by [5, Corollary 2.17] Λ is R-free (and hence R-reflexive) and gl.dim. $\Lambda=$gl.dim. R. Furthermore, since gl.dim. Λ $=$gl.dim. R and Λ is torsion-free over R, for any prime ideal $\mathfrak{Q}$ of R,

$$\text{gl.dim. } \Lambda \otimes_R R_{\mathfrak{Q}} = \text{gl.dim. } R_{\mathfrak{Q}}$$

[5, Corollary 3.4]. Thus our assertion is justified by the Auslander-Goldman criterion in dimension one. A direct proof that $\Lambda \otimes_R R_{\mathfrak{P}}$ is quasi-local seems difficult, and an example is provided which may suggest why. We exhibit a Λ which has global dimension three and is quasi-local. Furthermore, it is maximal. But for a certain height *two* prime $\mathfrak{P}$ of R, $\Lambda \otimes_R R_{\mathfrak{P}}$ is *not* quasi-local.

Preliminaries. All rings in this paper have units, and every module is unital and finitely generated. In particular, every algebra is a finitely generated module over its center.

We shall use the abbreviations gl.dim. for global dimension, p.d. for projective dimension, and codim for codimension. (If A is a commutative local ring and E is a nonzero A-module, then $\operatorname{codim}_A E$ is the length of the longest possible sequence of nonunits $x_1,\ldots,x_t$ in A such that x_1 is not a zero-divisor on E and if $2\leqq i\leqq t$, x_i is not a zero-divisor on $E/(x_1,\ldots,x_{i-1})E$.) $(R, \mathfrak{M})$ will always be a regular local ring with maximal ideal $\mathfrak{M}$, and K will denote its quotient field. Σ is a central simple K-algebra.

1. R has arbitrary (finite) global dimension.

PROPOSITION 1.1. *Suppose Γ is a central R-algebra. If* gl.dim. $\Gamma<\infty$ *and Γ is quasi-local, then $\Gamma\otimes_R K$ is a central simple K-algebra.*

Proof. By [5, Corollary 2.17], gl.dim. $\Gamma=$gl.dim. R and Γ is R-free. Thus by [5, Corollary 3.4], gl.dim. $\Gamma \otimes_R K=$gl.dim. $K=0$. Hence $\Gamma \otimes_R K$ is semisimple, say $\Gamma \otimes_R K=\Sigma_1 \oplus\cdots\oplus \Sigma_s$ where each Σ_i is a simple K-algebra. Since the center of Γ is R, the center of $\Gamma \otimes_R K$ is K. If K_i is the center of Σ_i, then $K=K_1 \oplus\cdots\oplus K_s$. But K is a field, so $s=1$. Thus $\Gamma \otimes_R K=\Sigma_1$, which is a central simple K-algebra. ∎

PROPOSITION 1.2. *Let $\Omega\subset\Gamma$ be R-orders in Σ. Suppose* gl.dim. $\Omega=$gl.dim. $R=n$ *and Γ is R-free. Then* gl.dim. $\Gamma=n$.

Proof. Since Γ is R-free, gl.dim. Γ=gl.dim. R if and only if every R-free Γ-module E is Γ-projective [5, Proposition 3.5]. Since gl.dim. $\Omega=n$,

$$n \geqq \text{p.d.}_\Omega E/\mathfrak{M}E = \text{codim}_R E+\text{p.d.}_\Omega E = n+\text{p.d.}_\Omega E.$$

($\text{codim}_R E=\text{codim}_R R=n$ since E is R-free.) It follows that $\text{p.d.}_\Omega E=0$, i.e., E is Ω-projective. Now E is Γ-projective by [4, Lemma 1.3]. ∎

Corollary 1.3. *Suppose* gl.dim. $R=2=$gl.dim. Ω *where* Ω *is an R-order in* Σ. *Then if* Γ *is any maximal order containing* Ω, gl.dim. $\Gamma=2$.

Proof. Any maximal order is R-reflexive [1, Theorem 1.5]. Hence Γ is R-free, since gl.dim. $R=2$. Proposition 1.2 now applies. ∎

Theorem 1.4. *Suppose* Λ *is an R-order in* Σ *such that* gl.dim. $\Lambda<\infty$ *and* Λ *is quasi-local. If* Γ *is any R-free order in* Σ *containing* Λ, *then* $\Gamma=\Lambda$. *Consequently if* Λ *is contained in an R-free maximal order, then* Λ *is maximal.*

Proof. By [5, Corollary 2.17] gl.dim. $\Lambda=n=$gl.dim. R. Let

$$C^r_\Lambda(\Gamma) = \{x \in \Sigma \mid \Gamma x \subset \Lambda\},$$

the right conductor of Γ in Λ. Since $\Lambda\subset\Gamma$, $C^r_\Lambda(\Gamma)=\text{Hom}_\Lambda(\Gamma, \Lambda)$. Since Γ is R-free and gl.dim. $\Lambda=$gl.dim. R, Γ is Λ-projective by [5, Proposition 3.5] and therefore $\text{Hom}_\Lambda(\Gamma, \Lambda)$ is Λ-projective. Thus $C^r_\Lambda(\Gamma)$ is a projective right ideal in Λ, and since Λ is quasi-local, $C^r_\Lambda(\Gamma)$ is principal (see [1, Proposition 3.3]). Say $C^r_\Lambda(\Gamma)=t\Lambda$. $\Gamma t\subset C^r_\Lambda(\Gamma)$ so $\Gamma t\subset t\Lambda$. But t is a unit in Σ since $C^r_\Lambda(\Gamma)\cap R\neq(0)$. Hence $\Gamma\subset t\Lambda t^{-1}$. Since $\Lambda\subset\Gamma$, we have $\Lambda\subset t\Lambda t^{-1}$. We shall show that $\Lambda=t\Lambda t^{-1}$, from which it follows that $\Lambda=\Gamma$.

Let $\sigma:\Sigma\to\Sigma$ be the inner automorphism $x \rightsquigarrow txt^{-1}$. Thus $\Lambda\subset\sigma(\Lambda)$, which implies that $\sigma(\Lambda)\subset\sigma^2(\Lambda)$, and so on. We obtain an increasing chain of orders $\Lambda\subset\sigma(\Lambda)\subset\sigma^2(\Lambda)\subset\cdots\sigma^i(\Lambda)\cdots$. But there can be no infinite ascending chain of R-orders in Σ. For the union of such a chain would be a subring of Σ integral over R, and hence finitely generated as an R-module (see [3, p. 70, Satz 6]). But R is noetherian, so a finitely generated R-module cannot contain an infinite ascending chain of submodules. Thus for some $m\geqq 1$, $\sigma^m(\Lambda)=\sigma^{m+1}(\Lambda)$. Therefore $\sigma(\Lambda)=\Lambda$ and we are done. ∎

Remark. By Proposition 1.1 we could have replaced the hypothesis that Λ is an R-order in Σ with the hypothesis that R is the center of Λ. The same is true for Theorem 2.2 in the next section.

2. The global dimension of R is three.

We begin by stating a lemma which we have been able to prove only for regular local rings of dimensions at most three.

Lemma 2.1. *Let* Λ *be an R-algebra which as an R-module is R-free. Suppose* gl.dim. $\Lambda=$gl.dim. $R(=3)$. *Let E be a left* Λ*-module which is R-reflexive, and suppose that* $\text{End}_\Lambda(E)$ *is R-free. Then E is* Λ*-projective.*

Before proving this lemma let us state and demonstrate the main theorem.

THEOREM 2.2. *R is a regular local ring of dimension three. Let* Λ *be a quasi-local R-order in* Σ *with* gl.dim. $\Lambda<\infty$. *If there exists an R-free maximal order* Γ *in* Σ *then* $\Lambda=t^{-1}\Gamma t$ *for some unit t in* Σ. *Thus* Λ *is maximal and all R-free maximal orders in* Σ *are isomorphic.*

Proof. Since Γ is a maximal order and R is an integrally closed noetherian domain, for any R-order Ω in Σ, $\Gamma=\text{End}_{\Omega}(C_{\Omega}^{r}(\Gamma))$ [5, Theorem 6.2]. So $\Gamma=\text{End}_{\Lambda}(C_{\Lambda}^{r}(\Gamma))$. Since Γ is R-free (by [5, Corollary 2.17]) and hence R-reflexive, $C_{\Lambda}^{r}(\Gamma)$ is also R-reflexive [5, Proposition 5.3]. But Γ is R-free, so $C_{\Lambda}^{r}(\Gamma)$ is Λ-projective by Lemma 2.1. Just as in the proof of Theorem 1.4 we have $C_{\Lambda}^{r}(\Gamma)=t\Lambda$ for some t which is invertible in Σ. Again, since $C_{\Lambda}^{r}(\Gamma)$ is a left Γ-module, $\Gamma t\subset t\Lambda$ and so $\Gamma\subset t\Lambda t^{-1}$. But Γ is maximal, so $\Gamma=t\Lambda t^{-1}$ and thus $\Lambda=t^{-1}\Gamma t$. ∎

COROLLARY 2.3. *If* Σ *is a full matrix ring* $M_n(K)$ *and* Λ *is a quasi-local R-order in* Σ *and* gl.dim. $\Lambda<\infty$ *then* Λ *is maximal and* $\Lambda\simeq M_n(R)$.

Proof. $M_n(R)$ *is* an R-free maximal order in Σ.

PROPOSITION 2.4. *Let R be regular local of dimension three. Suppose* Λ *is an R-free maximal order and* gl.dim. $\Lambda=3$. *Let I be a two-sided ideal in* Λ *which is R-reflexive. Then I is R-free (and hence* Λ*-projective).*

Proof. Since I is two-sided, $\text{End}_{\Lambda}(I)$ is an R-order containing Λ. By the maximality of Λ, $\text{End}_{\Lambda}(I)=\Lambda$ and is thus R-free. Since I is R-reflexive and gl.dim. $\Lambda=3$, the desired result follows from Lemma 2.1. ∎

COROLLARY 2.5. *Let R and* Λ *satisfy the hypotheses of the preceding proposition. Suppose for* $i=1, 2$ *that* I_i *is a two-sided ideal in* Λ *and is* Λ*-projective. Then* $I_1\cap I_2$ *is* Λ*-projective.*

Proof. Since I_i is Λ-projective it is R-free and hence R-reflexive. Therefore $I_1\cap I_2$ is R-reflexive, since over an integrally closed noetherian domain the double dual of a finitely generated torsion-free module is the intersection of its localizations at all the height one primes, and for any prime $\mathfrak{P}$, we have $(I_1\cap I_2)_{\mathfrak{P}}=(I_1)_{\mathfrak{P}}\cap(I_2)_{\mathfrak{P}}$. Clearly $I_1\cap I_2$ is two-sided. By Proposition 2.4 $I_1\cap I_2$ is Λ-projective. ∎

We return now to the proof of Lemma 2.1. This lemma is a generalization of [1, Theorem 4.4 (b) ⇒ (a)]. The idea is to ape that proof, which consists of a sequence of propositions. We shall state each of those propositions in its generalized version. For the first two we omit proofs since the proofs of their counterparts in [1] carry over nearly verbatim.

Throughout this proof $(S, \mathfrak{M})$ will be a commutative noetherian local ring with maximal ideal $\mathfrak{M}$ and Γ will be an S-algebra which is a finitely generated S-module.

PROPOSITION 2.1 A (CF. [1, PROPOSITION 4.7]). *Let A and B be* Γ*-modules such that* $\text{Hom}_{\Gamma}(A, B)\neq 0$. *If* $\text{codim}_S B\geqq i$ *for* $i=1, 2$ *then* $\text{codim}_S \text{Hom}_{\Gamma}(A, B)\geqq i$.

PROPOSITION 2.1 B (CF. [1, LEMMA 4.8]). *Let S be regular local of dimension at least three and let B be a Γ-module such that* $\operatorname{codim}_S B \geqq 2$. *If A is a Γ-module such that* $\operatorname{Hom}_\Gamma(A, B)$ *is S-projective and* $\operatorname{Ext}^1_\Gamma(A, B) \neq 0$, *then* $\operatorname{codim}_S \operatorname{Ext}^1_\Gamma(A, B) > 0$.

PROPOSITION 2.1 C (CF. [1, PROPOSITION 4.9]). *If S is a regular local ring, Γ is S-free*, gl.dim. $\Gamma_{\mathfrak{P}} =$ gl.dim. $S_{\mathfrak{P}}$ *for all primes $\mathfrak{P}$ of height at most two, E a Γ-module which is S-reflexive and* $\operatorname{Hom}_\Gamma(E, E)$ *is S-free, then* $\operatorname{Ext}^1_\Gamma(E, E) = 0$.

Proof. By induction on gl.dim. S. If gl.dim. $S \leqq 2$, then since E is S-reflexive it is S-projective. By hypothesis Γ is S-free and gl.dim. $\Gamma =$ gl.dim. S (since we are now assuming that the height of $\mathfrak{M}$ is at most two). Thus E is Γ-projective by [5, Proposition 3.5], and so $\operatorname{Ext}^1_\Gamma(E, E) = 0$.

Now suppose that gl.dim. $S = k \geqq 3$ and the proposition is true for rings of dimension less than k. Let $\mathfrak{P}$ be a nonmaximal prime ideal of S. Then gl.dim. $S_{\mathfrak{P}} < k$. $E_{\mathfrak{P}}$ is $S_{\mathfrak{P}}$-reflexive and $\operatorname{Hom}_{S_{\mathfrak{P}}}(E_{\mathfrak{P}}, E_{\mathfrak{P}}) = \operatorname{Hom}_S(E, E) \otimes_S S_{\mathfrak{P}}$ is $S_{\mathfrak{P}}$-free. Any prime ideal of $S_{\mathfrak{P}}$ of height at most two can be represented as $q_{\mathfrak{P}}$ where q is a prime ideal of S of height at most two. $(\Gamma_{\mathfrak{P}})_{q_{\mathfrak{P}}} = \Gamma_q$ and $(S_{\mathfrak{P}})_{q_{\mathfrak{P}}} = S_q$ so that gl.dim. $(\Gamma_{\mathfrak{P}})_{q_{\mathfrak{P}}}$ $=$ gl.dim. $(S_{\mathfrak{P}})_{q_{\mathfrak{P}}}$. Hence by our induction hypothesis $\operatorname{Ext}^1_{\Gamma_{\mathfrak{P}}}(E_{\mathfrak{P}}, E_{\mathfrak{P}}) = 0$, i.e., $\operatorname{Ext}^1_\Gamma(E, E) \otimes_S S_{\mathfrak{P}} = 0$. Thus no nonmaximal prime of S is an associated prime of $\operatorname{Ext}^1_\Gamma(E, E)$, so if $\operatorname{Ext}^1_\Gamma(E, E) \neq 0$, then $\operatorname{Ass}_S(\operatorname{Ext}^1_\Gamma(E, E)) = \{\mathfrak{M}\}$. But then $\operatorname{codim}_S \operatorname{Ext}^1_\Gamma(E, E) = 0$. By Proposition 2.1 B this is impossible, so $\operatorname{Ext}^1_\Gamma(E, E) = 0$. ∎

PROPOSITION 2.1 D (CF. [1, PROPOSITION 4.10]). *Suppose the S-algebra Γ is quasi-local and E is a Γ-module with* p.d.$_\Gamma$ $E = n$. *If A is a nonzero Γ-module then* $\operatorname{Ext}^n_\Gamma(E, A) \neq 0$.

Proof. Since p.d.$_\Gamma$ $E = n$, $\operatorname{Ext}^n_\Gamma(E, T) \neq 0$ for some simple Γ-module T [5, Corollary 1.5]. Since Γ is quasi-local, all simple Γ-modules are isomorphic. It follows that A/JA is a direct sum of copies of T, where $J = \operatorname{Rad} \Gamma$. Ext commutes with direct sums, so $\operatorname{Ext}^n_\Gamma(E, A/JA) \neq 0$. From the exact sequence

$$0 \to JA \to A \to A/JA \to 0$$

we obtain a long exact sequence

$$\cdots \operatorname{Ext}^n_\Gamma(E, A) \to \operatorname{Ext}^n_\Gamma(E, A/JA) \to \operatorname{Ext}^{n+1}_\Gamma(E, JA) \cdots.$$

$\operatorname{Ext}^{n+1}_\Gamma(E, -) = 0$ since p.d.$_\Gamma$ $E = n$. It follows that $\operatorname{Ext}^n_\Gamma(E, A) \neq 0$. ∎

We may now complete the proof of Lemma 2.1. Since gl.dim. $R = 3 = \operatorname{codim}_R R$, $\operatorname{codim}_R \operatorname{Hom}_R(E^*, R) \geqq 2$ by [1, Proposition 4.7] (X^* denotes $\operatorname{Hom}_R(X, R)$). But $\operatorname{Hom}_R(E^*, R) = E^{**}$ and by hypothesis $E = E^{**}$, so $\operatorname{codim}_R E \geqq 2$. Hence p.d.$_R$ $E \leqq 1$. Since Λ is R-free and gl.dim. $\Lambda =$ gl.dim. R, every R-free Λ-module is Λ-projective [5, Proposition 3.5]. From this it follows that for any Λ-module A, p.d.$_\Lambda$ A $=$ p.d.$_R$ A. Thus p.d.$_\Lambda$ $E \leqq 1$. But since $\operatorname{End}_\Lambda(E)$ is R-free, $\operatorname{Ext}^1_\Lambda(E, E) = 0$ by Proposition 2.1 C. (Since gl.dim. $\Lambda =$ gl.dim. R, for any prime $\mathfrak{P}$ in R gl.dim. $\Lambda_{\mathfrak{P}}$ $=$ gl.dim. $R_{\mathfrak{P}}$ by [5, Corollary 3.4].) Now by Proposition 2.1 D p.d.$_\Lambda$ E cannot be 1, so it must be 0 and E is Λ-projective. ∎

3. **More practical sufficient conditions.** We shall give some more practical sufficient conditions for the maximality of R-orders having finite global dimension.

THEOREM 3.1. *Let R be regular local of dimension n. Assume that the R-order Λ is R-reflexive. Suppose that for some $x \in \mathfrak{M} - \mathfrak{M}^2$, $\Lambda_{(x)}$ is $R_{(x)}$-separable and $\Lambda/x\Lambda$ is a maximal R/x-order in $\Lambda(x)/x\Lambda_{(x)}$.*

Then Λ is maximal.

Proof. It suffices, by [1, Theorem 1.5], to show that for every height one prime $\mathfrak{Q}$ in R, $\Lambda_{\mathfrak{Q}}$ is maximal.

If $\mathfrak{Q}=(x)$ we are done since $\Lambda_{(x)}$ is $R_{(x)}$-separable and therefore maximal [2, Proposition 7.2]. Suppose $\mathfrak{Q}\neq(x)$. Let $\mathfrak{P}$ be a minimal associated prime of $xR+\mathfrak{Q}$. Then height $\mathfrak{P} \leqq 1+\text{height } \mathfrak{Q}=2$. Since $\mathfrak{Q}\neq(x)$ and height $\mathfrak{Q}=1$, $x \notin \mathfrak{Q}$. Therefore $\mathfrak{P}$ properly contains $\mathfrak{Q}$, and so height $\mathfrak{P}=2$. $\mathfrak{P}/x$ is a height one prime in R/x. Since $\Lambda/x\Lambda$ is maximal, $(\Lambda/x\Lambda)_{\mathfrak{P}/x}$ is maximal over a discrete valuation ring and is therefore quasi-local. But $(\Lambda/x\Lambda)_{\mathfrak{P}/x} \simeq \Lambda_{\mathfrak{P}}/x\Lambda_{\mathfrak{P}}$, so $\Lambda_{\mathfrak{P}}/x\Lambda_{\mathfrak{P}}$ is hereditary and quasi-local. Also, since x is regular on $\Lambda_{\mathfrak{P}}$ and gl.dim. $\Lambda_{\mathfrak{P}}/x\Lambda_{\mathfrak{P}}<\infty$, by [5, Proposition 5.6] gl.dim. $\Lambda_{\mathfrak{P}}=1+\text{gl.dim. } \Lambda_{\mathfrak{P}}/x\Lambda_{\mathfrak{P}}=2$, and so $\Lambda_{\mathfrak{P}}$ is maximal by [5, Theorem 5.4]. Hence $(\Lambda_{\mathfrak{P}})_{\mathfrak{Q}_{\mathfrak{P}}}$ is maximal, i.e. $\Lambda_{\mathfrak{Q}}$ is maximal. ∎

COROLLARY 3.2. *Let R be regular local and let Λ be a maximal R-order in the central simple K-algebra Σ. Let $\Lambda[[X]]=\Lambda \otimes_R R[[X]]$ and $\Sigma((X))=\Sigma \otimes_K K((X))$, where $R[[X]]$ denotes the formal power series ring over R in one indeterminate and $K((X))$ is its quotient field. Then $\Lambda[[X]]$ is a maximal $R[[X]]$-order in the central simple $K((X))$-algebra $\Sigma((X))$.*

Proof. $\Sigma((X))$ is central separable (and hence central simple) over $K((X))$ by [2, Corollary 1.6]. To simplify notation, let $S=R[[X]]$ and $\Gamma=\Lambda[[X]]$. Λ is R-reflexive and S is R-free and hence R-flat, so Γ is S-reflexive. $S/XS\simeq R$ and $\Gamma/X\Gamma\simeq\Lambda$. gl.dim. $S=1+\text{gl.dim. } R$, so S is regular local. $\Gamma_{(X)}/X\Gamma_{(X)}\simeq\Lambda \otimes_R K=\Sigma$. So $\Gamma_{(X)}/X\Gamma_{(X)}$ is central separable over $S_{(X)}/XS_{(X)}$ and thus $\Gamma_{(X)}$ is central separable over $S_{(X)}$. By our hypothesis on Λ, $\Gamma/X\Gamma$ is a maximal S/XS-order in $\Gamma_{(X)}/X\Gamma_{(X)}$. The maximality of Γ now follows from Theorem 3.1. ∎

COROLLARY 3.3. *Suppose R is regular local of dimension three. Assume that Λ is quasi-local,* gl.dim. $\Lambda<\infty$, *and $\mathfrak{M}\not\subseteq N^2$, where $N=$*Rad Λ. *Then Λ is maximal.*

Proof. Λ is R-free (and hence R-reflexive) by [5, Corollary 2.17]. Since $\mathfrak{M}\not\subseteq N^2$ there is an $x \in \mathfrak{M}-N^2$ such that $\Lambda_{(x)}$ is $R_{(x)}$-separable. (For a proof of this, see the proof of [5, Theorem 5.7].) Furthermore, $\Lambda/x\Lambda$ is quasi-local, and since $x \in \mathfrak{M}-N^2$ and gl.dim. $\Lambda<\infty$, gl.dim. $\Lambda/x\Lambda=\text{gl.dim. } \Lambda-1$ [5, Theorem 6.9]. It follows from [5, Theorem 5.4] that $\Lambda/x\Lambda$ is maximal in $\Lambda_{(x)}/x\Lambda_{(x)}$. The preceding theorem now applies. ∎

EXAMPLE 1. (a) Let R be regular local of dimension 2 and suppose X and Y generate the maximal ideal $\mathfrak{M}$. Define a quaternion algebra $\Gamma=R[1, \sigma, \tau, \sigma\tau]$ (i.e.,

Γ is R-free with generators $1, \sigma, \tau, \sigma\tau$) by setting $\sigma^2 = X$, $\tau^2 = Y$, and $\tau\sigma = -\sigma\tau$. Then Rad $\Gamma = \Gamma(\sigma, \tau)$ and $\Gamma/\text{Rad } \Gamma \approx R/\mathfrak{M}$. Thus Γ is local. Define

$$f: \Gamma \oplus \Gamma \to \Gamma(\sigma, \tau) \quad \text{by } f(\gamma_1, \gamma_2) = \gamma_1\sigma + \gamma_2\tau.$$

$$0 \longrightarrow \Gamma \cdot \langle \tau, \sigma \rangle \longrightarrow \Gamma \oplus \Gamma \xrightarrow{f} \Gamma(\sigma, \tau) \longrightarrow 0$$

is then a Γ-free resolution of Rad Γ. Thus p.d._Γ Rad $\Gamma \leqq 1$. gl.dim. $\Gamma = \text{p.d.}_\Gamma\, \Gamma/\text{Rad } \Gamma$ $= 1 + \text{p.d.}_\Gamma$ Rad $\Gamma \leqq 2$. Since Γ is local, gl.dim. Γ = gl.dim. $R = 2$.

(b) Let R be regular local of dimension three and assume that 2 is a unit in R. Suppose $\mathfrak{M} = (X, Y, Z)$. Define a quaternion algebra $\Lambda = R[1, \alpha, \beta, \alpha\beta]$ by setting $\alpha^2 = X + Z^2$, $\beta^2 = Y$, and $\alpha\beta = -\beta\alpha$. Now $\Lambda/Z\Lambda$ is isomorphic to the algebra Γ in part (a). So gl.dim. $\Lambda/Z\Lambda = 2$ and $\Lambda/Z\Lambda$ is local. Thus Λ is local, and by [5, Proposition 5.6] gl.dim. $\Lambda = 3$. $N = \text{Rad } \Lambda = \Lambda(\alpha, \beta, Z)$. $Z \notin N^2$, so $\mathfrak{M} \not\subset N^2$. By Corollary 3.3 Λ is maximal.

Λ is rather strange, though. It itself is local, and since it is maximal, Λ_q is quasi-local for every height one prime q of R. However, $\mathfrak{P} = (X, Y)R$ is a height two prime, such that $\Lambda_{\mathfrak{P}}$ is *not* quasi-local. It is easily seen that $\Lambda_{\mathfrak{P}}(\alpha - Z, X, \beta)$ and $\Lambda_{\mathfrak{P}}(\alpha + Z, X, \beta)$ are maximal two-sided ideals in $\Lambda_{\mathfrak{P}}$; indeed $\Lambda_{\mathfrak{P}}$ modulo either ideal is isomorphic to the field $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$. The two are distinct since $\alpha + Z - (\alpha - Z) = 2Z$ is a unit in $R_{\mathfrak{P}}$.

COROLLARY 3.4. *Let R be regular local of dimension three and Λ a quasi-local central R-algebra. Suppose that, for some $x \in \mathfrak{M} - \mathfrak{M}^2$,* gl.dim. $\Lambda/x\Lambda < \infty$ *and the center of $\Lambda/x\Lambda$ is R/x. Then* gl.dim. $\Lambda = 3$ *and Λ is a maximal order in the central simple K-algebra $\Lambda \otimes_R K$.*

Proof. By Proposition 1.1, $\Lambda/x\Lambda \otimes_{R/x} R_{(x)}/xR_{(x)} = \Lambda_{(x)}/x\Lambda_{(x)}$ is central simple (or equivalently, central separable) over $R_{(x)}/xR_{(x)}$. Hence $\Lambda_{(x)}$ is $R_{(x)}$-separable [2, Theorem 4.7].

$\Lambda/x\Lambda$ is quasi-local, has finite global dimension and gl.dim. $R/x = 2$, so by [5, Theorem 5.4] $\Lambda/x\Lambda$ is a maximal R/x-order in $\Lambda_{(x)}/x\Lambda_{(x)}$, and gl.dim. $\Lambda/x\Lambda = 2$. By [5, Proposition 5.6] gl.dim. Λ = gl.dim. $\Lambda/x\Lambda + 1 = 3$. It follows from Proposition 1.1 that $\Lambda \otimes_R K$ is a central simple K-algebra. Since Λ is quasi-local and gl.dim. $\Lambda < \infty$, Λ is R-free. Hence by Theorem 3.1, Λ is maximal in $\Lambda \otimes_R K$. ∎

We conclude with an example for which Corollary 3.4, but not Corollary 3.3, is applicable.

EXAMPLE 2. Let R be regular local of dimension three, $\mathfrak{M} = (X, Y, Z)$, and characteristic $(R/\mathfrak{M}) \neq 2$. Let $\Lambda = R[1, \alpha, \beta, \alpha\beta]$ where $\alpha^2 = X$, $\beta^2 = Y$, and $\beta\alpha = -\alpha\beta + Z$.

Then $N = \text{Rad } \Lambda = \Lambda(\alpha, \beta)$. Notice that $Z \in N^2$, so that $\mathfrak{M} \subset N^2$. Thus Corollary 3.3 does not apply. However, consider $\Lambda/Z\Lambda$. It is $R/Z[1, \bar\alpha, \bar\beta, \bar\alpha\bar\beta]$ where $\bar\alpha^2 = X$, $\bar\beta^2 = Y$, and $\bar\beta\bar\alpha = -\bar\alpha\bar\beta$. This is the ring Γ in part (a) of Example 1. Hence $\Lambda/Z\Lambda$ is local and gl.dim. $\Lambda/Z\Lambda=2$. Since characteristic $(R/\mathfrak{M})\neq 2$, 2 is a unit in R/Z and therefore the center of $\Lambda/Z\Lambda$ is R/Z. From Corollary 3.4 it follows that Λ is maximal.

References

1. M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc. **97** (1960), 1–24. MR **22** #8034.

2. ———, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367–409. MR **22** #12130.

3. M. Deuring, *Algebren*, Springer, Berlin, 1935.

4. M. Harada, *Hereditary orders*, Trans. Amer. Math. Soc. **107** (1963), 273–290. MR **27** #1474.

5. M. Ramras, *Maximal orders over regular local rings of dimension two*, Trans. Amer. Math. Soc. **142** (1969), 457–479. MR **39** #6878.

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